INVERSE PROBLEM OF HEAT CONDUCTION

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UDC 536.24.02

We consider the one-dimensional inverse problem of nonstationary heat conduction and we analyze the physical and mathematical causes of instability in various forms of its solution.

For the heat conduction equation we can identify three types of inverse problems, which may be inferred in seeking the cause for a known effect:

1) determination of the boundary conditions,

2) determination of the coefficients of the equation,

3) determination of the temperature field for a time in the past.

In this paper we investigate a problem of the first type [1-9, 15-23] in which, from the results of measurements of temperature in the interior of a body, it is required to recover the heat flow or temperature on its surface as a function of the time (a problem of this type was considered for the first time in [2, 15]). In the majority of cases this is the only way to determine the thermal boundary conditions when studying heat and mass transfer processes experimentally. This problem takes on a special value in the study of nonstationary heat phenomena.

A basic problem which arises when solving problems of this kind is the difficulty of obtaining sufficiently exact and sufficiently stable results. Overcoming these difficulties must be based on a careful study of the characteristic features of inverse problems and a strict justification for the choice of mathematical methods for their solution.

We consider the heat conduction process in an infinite plate, described by the equation

$$C\frac{\partial T}{\partial \tau} = \frac{\partial}{\partial x}\left(\lambda \frac{\partial T}{\partial x}\right),\quad 0 < x < b,\ \tau > 0. \tag{1}$$

At the initial instant the distribution $T(x,\ 0) = \varphi(x)$ is known. A temperature transmitter is placed at the point $x = x_1$ ($x_1 > 0$).

We assign the following conditions on the boundaries of the region $x_1 \leq x \leq b$:

$$\begin{gathered} T(x_1,\ \tau) = T_1(\tau), \\ T(b,\ \tau) = T_i(\tau) \quad \text{or} \quad -\lambda\frac{\partial T(b,\ \tau)}{\partial x} = q_i(\tau) \end{gathered}, \tag{2}$$

where $T_1(\tau)$, $T_i(\tau)$, and $q_i(\tau)$ are known functions.

The Eqs. (1) and (2) correspond to the direct problem of heat conduction. Solving it, we determine the heat flow at the boundary $x = x_1$: $q_1(\tau) = -\lambda[\partial T(x_1,\ \tau)]/[\partial x]$.

In the region $0 \leq x \leq x_1$ we consider the Cauchy problem corresponding to the continuation of the solution of the heat conduction equation up to the boundary $x = 0$ of the region from the known conditions at the point x_1:

$$\begin{gathered} T(x_1,\ \tau) = T_1(\tau), \\ -\lambda\frac{\partial T(x_1,\ \tau)}{\partial x} = q_1(\tau). \end{gathered} \tag{3}$$

Translated from Inzhenerno-Fizicheskii Zhurnal, Vol. 25, No. 3, pp. 530-537, September, 1973. Original article submitted September 8, 1971.

Equations (1) and (3) define the inverse problem of heat conduction from whose solution we wish to determine the surface temperature $T_w(\tau) = T(0, \tau)$ or the heat flow into the body, namely, $q(\tau) = -\lambda[\partial T(0, \tau)]/[\partial x]$. We remark that it is not necessary to reduce the inverse problem to a Cauchy problem and that another formulation of the problem is possible in which the solution is sought directly from the conditions given on the boundaries $x = x_1$ and $x = b$.

By convention, we represent the inverse heat conduction problem by the first order operator equation

$$Au = f,\ u \in D_A,\ f \in Q_A, \tag{4}$$

where A is a continuous operator acting from the space of solutions U into the space of the input data F; u is the solution sought; f denotes input data; $D_A \subseteq U$ is the domain of definition of the operator A; $Q_A = A(D_A) \subseteq F$ is the domain of the values of the operator A.

In the general case the problem of solving the given equation is considered to be well-posed if its solution satisfies the following requirements:

1) it exists for arbitrary $f \in Q_A = F$,

2) it is unique in U,

3) it is stable, i.e., to small variations of the right hand side f in the metric of the space F there correspond small deviations of the solution u in the metric of the space U.

Henceforth we assume that the solution (4) exists and is unique. We base this assumption on the physics of the inverse heat conduction problem, an assumption which is corroborated by Kovalevskaya's theorem [10], according to which the solution of the Cauchy problem exists and is unique providing that $T(x)$, $q_1(\tau)$, and the solution sought are analytic.

We pause to analyze the third of the requirements stated above for the problem to be well-posed. For this purpose we consider the possible direct methods for solving the inverse problem in a linear formulation (λ = const, $a = \lambda/C$ = const). By direct methods we mean methods of solving the initial problem directly by establishing the inverse correspondence: $u = R(f)$. The papers [1-9, 15-18, 22, 23] are concerned with this aspect of the problem.

Solution in the Form of a Power Series. We assume that the functions $T_1(\tau)$ and $q_1(\tau)$ are infinitely differentiable ($T_1^{(n)}(\tau) = d^nT_1/d\tau^n$, $q_1^{(n)}(\tau) \equiv d^nq_1/d\tau^n$, $n \to \infty$) and that the coefficients in the heat conduction equation are constants. In this case we can write down formally the Stefan solution for determining the temperature field; thus

$$T(x, \tau) = \sum_{n=0}^{\infty} \left[\frac{T_1^{(n)}(\tau)(x - x_1)^{2n}}{(2n)!\, a^n} - \frac{q_1^{(n)}(\tau)(x - x_1)^{2n+1}}{(2n+1)!\, a^n \lambda} \right]. \tag{5}$$

Differentiating the relation (5) with respect to x and retaining the first N terms, we write down a finite expression for the heat flow on the boundary of the body at x = 0:

$$q(\tau) = -\sum_{n=0}^{N-1} \frac{q_1^{(n)} x_1^{2n}}{(2n)!\, a^n} + \lambda \sum_{n=1}^{N} \frac{T_1^{(n)} x_1^{2n-1}}{(2n-1)!\, a^n}. \tag{6}$$

For the temperature of the surface of the body we have

$$T_w(\tau) = \sum_{n=0}^{N-1} \left[\frac{T_1^{(n)}(\tau)\, x_1^{2n}}{(2n)!\, a^n} + \frac{q_1^{(n)}(\tau)\, x_1^{2n+1}}{(2n+1)!\, a^n \lambda} \right]. \tag{7}$$

Thus, in determining the boundary conditions in accordance with the given form of the solution of the inverse heat conduction problem, it is necessary to calculate the values of the derivatives up to the N-th order. Since the functions T_1 and q_1 are usually given discretely with certain fluctuating errors and the operator of differentiation represents a typical case of unbounded operators, the direct method for solving the inverse problem, indicated above, leads in the general case to an unstable calculation process.

Integral Form of the Inverse Heat Conduction Problem. In the linear formulation inverse problems reduce to the solution of Volterra type integral equations of the first kind:

$$Au \equiv \int_{\tau_0}^{\tau} K(\tau, \xi)\, u(\xi)\, d\xi = f(\tau), \quad \tau_0 \leqslant \tau,\ \xi \leqslant \tau_m. \tag{8}$$

A simple and widely used method for solving Eq. (8) involves reducing it to an equation of the second kind through the operation of differentiation:

$$u(\tau) + \int_{\tau_0}^{\tau} \frac{K_\tau(\tau, \xi)}{K(\tau, \tau)} u(\xi)\, d\xi = \frac{f'(\tau)}{K(\tau, \tau)}.$$

However, such a procedure is possible if the kernel of the original equation nowhere vanishes on the interval $[\tau_0, \tau_m]$ ($K(\tau, \tau) \neq 0$). In our case

$$K(\tau - \xi) \to 0, \quad \xi \to \tau,$$

so that we cannot apply this method for solving the integral equation (8). In addition, repeated differentiation also does not give the desired result. Consequently, it is necessary to solve the original Eq. (8) of the first kind directly.

The solution of this equation for $u(\tau)$ constitutes a typical ill-posed problem since the Volterra operator represents a completely continuous operator.

Finite-Difference Form of the Solution. If we approximate the derivatives in the expressions (6) and (7) by finite differences, we obtain computational relationships analogous to the solution of the linear inverse heat conduction problem in an explicit scheme difference form, where N is the number of layers on the segment $0 < x < x_1$ (see [8]). Thus the numerical method of solving the original problem will also be disposed towards an unstable computation.

Reduction to Variational Problems. It is desirable in the general case to carry through the recovery of the boundary conditions in such a way as to reduce the residual $\Delta = \rho_F(Au, f)$, corresponding to the deviation of the left side from the right side in the metric of the space F, to some value determined by the error in the approximate data of the problem. If the approximating operator $A_h = A$ and the input information f are known with accuracy $\delta = 0$, it is then necessary to have $\Delta \sim 0$. The method, which suggests itself, of obtaining the desired approximations from the solution of the problem of minimizing a quadratic functional can give rise to serious difficulties from a numerical standpoint. This is explained, on the one hand, by the fact that convergence with respect to the functional does not imply convergence of the approximate solutions to the true solution, and, on the other hand, by the fact that it is usually necessary to solve the problem with approximate data. The resulting "solutions" may differ substantially from the actual solutions.

Such a situation is the result of an ill-posed problem $Au = f$. The condition of the variational problem may in many cases prove to be worse than that of the original problem. Thus for the minimization of a quadratic functional use is often made of the Euler equation

$$A^*Au - A^*f = 0,$$

which is more poorly conditioned than $Au - f = 0$.

In the case of a linear law of heat conduction and a normalization corresponding to the space L_2 the functional has the form

$$\Delta = \left\{ \int_{\tau_0}^{\tau_m} \left[\int_{\tau_0}^{\tau} K(\tau, \xi)\, u(\xi)\, d\xi - f(\tau) \right]^2 d\tau \right\}^{\frac{1}{2}}.$$

Its minimization corresponds to the solution of a Fredholm integral equation of the first kind

$$\int_{\tau_0}^{\tau_m} \overline{K}(\xi, \zeta)\, u(\zeta)\, d\zeta = \overline{b}(\xi).$$

Such a problem is automatically ill-posed. Therefore the method of least squares, used in [3, 6-8] for recovering the heat flow, does not yield stable solutions for sufficiently small time steps $\Delta\tau$.

Thus from an analysis of the possible direct methods of solving the original inverse heat conduction problem we have established that these problems are mathematically ill-posed, there being an absence of a continuous dependence of the results on the input data.

We pause to consider the physical side of this phenomenon. An essential point in understanding the physical nature of the instability of inverse heat conduction problems is the strong smoothing of the characteristic singularities of the functions $T_w(\tau)$ and $q(\tau)$ as the temperature transmitter is moved deeper into the body from the heated surface. Therefore, to small deviations in f there will correspond large deviations in the unknown solution u. Since changes in the input data are accomplished with the aid of a specific system of physical instruments, these data are always known with a certain approximation δ, and the solution of the inverse heat conduction problem, corresponding to the measured input data, may differ substantially from the solution sought and may clearly display a pronounced oscillatory behavior. The representation of the original operator A in terms of the approximating operator A_h (with an accuracy of approximation h) plays a definite role in distortion of the results as do also the round-off errors in the computation. Moreover, there is an initial time interval $\Delta\tau_0$, determined by the threshold sensitivity of the measuring system, during which the instruments do not register in spite of the fact that heat is being applied to the body. This initial indeterminateness in the boundary condition will be present over the whole time period during which the investigation is taking place.

Thus, since in real experiments there is always an initial indeterminacy in the boundary conditions, the results of measurements are known with a certain error, and the main thing then is that small perturbations of the input data yield large deviations in the boundary conditions; therefore it is often not possible to obtain solutions of inverse heat conduction problems with good accuracy by means of direct methods. This conclusion does not apply to those cases wherein, for a given temperature $T(x_1, \tau) = T_W(\tau)$ of the heated wall ($x_1 = 0$), a recovery is made of the unknown heat flow $q(\tau)$ in the body. This is a correctly posed problem.

By way of example, we show this for the integral equation of the inverse heat conduction problem in the case of a semiinfinite body, namely,

$$\int_0^{\tau} K(\tau, \xi)\, q(\xi)\, d\xi = T(\tau), \tag{9}$$

where

$$K(\tau, \xi) = \frac{1}{\lambda_0}\sqrt{\frac{a}{\pi}}\,\frac{\exp\left[-\dfrac{x_1^2}{4a(\tau-\xi)}\right]}{\sqrt{\tau-\xi}}.$$

For the proof we use the method due to Holmgren (see [12]). We multiply both sides of Eq. (9) by $(z-\tau)^{-1/2}$ and integrate with respect to τ from 0 to z:

$$\int_0^z d\tau \int_0^{\tau} (z-\tau)^{-\frac{1}{2}} K(\tau, \xi)\, q(\xi)\, d\xi = \int_0^z (z-\tau)^{-\frac{1}{2}}\, T(\tau)\, d\tau.$$

We interchange the order of integration over the triangular region:

$$\int_0^z q(\xi)\, V(\xi, z)\, d\xi = \int_0^{\tau} (z-\tau)^{-\frac{1}{2}}\, T(\tau)\, d\tau, \tag{10}$$

where

$$V(\xi, z) = \int_{\xi}^{z} K(\tau, \xi)(z-\tau)^{-\frac{1}{2}}\, d\tau. \tag{11}$$

We differentiate the expression (10) with respect to z. The derivative of the right hand side of this expression has a meaning if the function $T(\tau)$ satisfies the Hölder condition

$$|T(\tau') - T(\tau'')| \leqslant K\,|\tau' - \tau''|^{\alpha}, \quad \left(\frac{1}{2} < \alpha \leqslant 1\right), \tag{12}$$

where K is a positive constant. It is defined, in this case, by the expression (see [12])

$$\frac{d}{dz}\int_0^{\tau} (z-\tau)^{-\frac{1}{2}}\, T(\tau)\, d\tau = z^{-\frac{1}{2}}\, T(z) - \frac{1}{2}\int_0^z (z-\xi)^{-\frac{3}{2}}\big(T(\xi) - T(z)\big)\, d\xi.$$

As a result we have

$$q(z)\, V(z, z) + \int_0^z q(\xi)\,\frac{\partial V(\xi, z)}{\partial z}\, d\xi = z^{-\frac{1}{2}}\, T(z) - \frac{1}{2}\int_0^z (z-\xi)^{-\frac{3}{2}}\big(T(\xi) - T(z)\big)\, d\xi. \tag{13}$$

It is not difficult to show that for x = 0 the quantity $V(\xi, z) = (1/\lambda_0)\sqrt{a\pi}$.

Replacing z by τ in the expression (13), we obtain, finally,

$$q(\tau)=\frac{\lambda}{\sqrt{a\pi}}\left[\frac{T_w(\tau)}{\sqrt{\tau}}-\frac{1}{2}\int_0^{\tau}\frac{T_w(\xi)-T_w(\tau)}{(\tau-\xi)^{\frac{3}{2}}}d\xi\right]. \tag{14}$$

If the condition (12), which, as a rule, is not a restrictive condition, is satisfied, the integrand function in Eq. (14) has a weak singularity and, consequently, the given integral exists. This integral can be evaluated by the known rules for the approximate integration of functions with points of discontinuity.

Thus we have reduced the problem of solving the integral equation of the first kind (9) for determining the heat flow in a semiinfinite body from a known temperature on its boundary to the correctly-posed problem of evaluating the expression (14). The result obtained is explained by the fact that the problem in question is, in fact, not an inverse heat conduction problem since for the determination of $q(\tau)$ the extension of the solution of the heat conduction equation to the boundary of the region is not required. Actually, the direct heat conduction problem must be solved in accordance with the conditions given on two boundaries of the region with a recalculation of a boundary condition of the first kind on a boundary condition of the second kind (pseudo-inverse heat conduction problem).

In a number of practically important cases direct methods can be used for solving ill-posed inverse heat conduction problems. This is usually tied in with a specific choice of the relationship involving the calculational time step, the distance to a point where the temperature in the body is fixed, and the thermophysical properties of the body. In addition, a critical value of the time step is determined such that the approximate solution obtained is also sufficiently smooth if $\Delta\tau \geq \Delta\tau_{crit}$ (see, for example, [7, 8]).

It is natural that the solution of inverse heat conduction problems by direct methods proves to be unsatisfactory in many cases since, in this regard, the inverse problems may be formally regarded as correctly-posed. The use of large steps in a calculation to remove "oscillations" in the solution can lead to a substantial decrease in accuracy of the results. Besides, one often encounters problems in practice in which the entire time interval considered is comparable in magnitude with $\Delta\tau_{crit}$ or is even less than the critical value of the step.

In the general case the methods for solving inverse problems must take into account a violation of the stability condition implied in their formulation, i.e., it is necessary to consider these as incorrectly formulated problems.

A. N. Tikhonov has recently given a very general method for solving ill-posed problems (the method of regularization) [13, 14]. His method involves the notion of a regularizing algorithm (operator). A one-parameter operator R^{α}, acting from F to U, is called a regularizing operator if R^{α} is defined on all of F for arbitrary $\alpha > 0$, is continuous on F, and, for arbitrary $u \in D$ (D some set in U), $\lim\limits_{\alpha\to 0} R^{\alpha}Au = u$.

The problems involved in constructing regularizers R^{α} for inverse heat conduction problems were considered in [19, 20, 21].

NOTATION

A	is the operator;
a	is the thermal diffusivity;
b	is the plate thickness;
C	is the volumetric heat capacity;
f	is the inlet data;
q	is the heat flux;
T_w, T_i	are the temperatures at external and internal walls, respectively;
u	is the unknown solution;
x	is the coordinate;
λ	is the thermal conductivity;
τ	is the time;
q_i	is the heat flux on internal wall;
T	is the temperature.

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